Low-Thrust Station Keeping Guidance for a 24-Hour Satellite

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The equations of motion for a nearly circular equatorial orbit, with inclusion of the J_2 and $J_2^{(2)}$ gravitational terms and of radial and tangential low-thrust forces, are solved by the method of variation of parameters. These nonlinear variation of parameters equations are analytic in a parameter proportional to the low-thrust acceleration level. Perturbations through second order in this parameter are computed in closed form by quadratures. A finite vector equation which is transcendental and connects the initial perturbation state and the unknown control sequence with the desired final state, is solved by the Newton-Raphson method. The solution consists of the constant thrust levels and the switching times that maximize the lifetime, for a given station longitude and longitude tolerance. Since the computations can be repeated on the basis of orbit determination data, this station keeping guidance scheme can be implemented in discrete closed-loop fashion. Numerical results show that relatively small initial errors in position and velocity cause longitude secular errors that require compensation by additional low thrusting. This demonstrates that a low-thrust control law that ignores the effect of initial condition errors overestimates the life time of a low-thrust synchronous satellite.

I. Introduction

A STATION keeping guidance scheme for a low-thrust gravity-gradient stabilized 24-hr satellite is presented. The thrust force can be switched in direction and to several magnitude levels that are piecewise constant in time.

The equations of motion for a nearly circular equatorial orbit, with inclusion of the J_2 and $J_2^{(2)}$ gravitational terms and of radial and tangential thrust components, are solved by the method of variation of parameters. Since the variational equation of motion in the binormal direction is decoupled, the perturbation state vector is reduced to one of four dimensions. The guidance analysis neglects the effects of luni-solar gravitational perturbations and solar radiation effects. The two-body problem solution of Fang and Brown[3] is used as the reference or intermediary orbit.

This solution was obtained by application of the method of Linstedt and Poincaré.[4] It provides for the dependence of the period of the solution, known to be periodic, on the initial values of the perturbed state. The domain of validity of the Fang and Brown solution is commensurate with the long time of application of the low thrust forces. The resulting nonlinear variation of parameters equations are thus analytic in a parameter $\gamma = A_0/R\omega^2$, where A_0 is a reference thrust acceleration, R is the synchronous radius, and ω is the Earth's spin rate.

Perturbations through second order in this parameter, proportional to the low-thrust acceleration level, are computed in closed form by quadratures. The motion consists of short period diurnal oscillations about a smooth motion that connects the states sampled in every revolution. The linear constant coefficient system that corresponds to the sampled perturbation state is not completely controllable. Nonetheless, the longitude error and the longitude-rate error, can be regulated to zero or to any values in a neighborhood of zero. Therefore the optimal control sequence for the minimum fuel regulator problem is not uniquely defined as having the bang-off-bang structure.

An optimal switching technique is derived by solving a simple extremal problem with a hard constraint on the final longitude and longitude rate.

The first-order solution to the thrust switching sequence is obtained in closed form expressions, accurate to order γ, using Cramer's rule. The first-order solution is taken as the initial guess to the Newton-Raphson solution of a transition vector equation, accurate to order γ^2 which connects the initial perturbation state and the unknown control sequence with the desired target state. The solution consists of the constant thrust levels and the switching times and is obtained from a double precision program, which uses numerical approximations to the required partial derivatives.

Numerical results are presented showing the dependence of satellite lifetime on station longitude, longitude tolerances, and initial position and velocity errors.

Since the computations can be rapidly repeated on the basis of orbit estimation data, this station keeping guidance scheme can be implemented in closed-loop fashion.

II. Equations of Motion

The equations of motion in an inertial equatorial plane with polar coordinates r, θ are

$$\ddot{r} - r\dot{\theta}^2 = -\mu/r^2 - \tfrac{3}{2}\mu J_2 R_e^2/r^4 + 9\mu J_2^{(2)}(R_e^2/r^4)\cos 2(\lambda - \lambda_2^{(2)}) + uf_r \quad (1)$$

$$(1/r)(d/dt)(r^2\dot{\theta}) = 6\mu J_2^{(2)}(R_e^2/r^4)\sin(\lambda - \lambda_2^{(2)}) + uf_\theta$$

Here the terms in $J_2^{(2)}$ and the thrust acceleration uf_r, uf_θ are considered as perturbations on the sidereal circular orbit of radius R characterized by the primary and J_2 field. Since on a sidereal circular orbit $\ddot{r} = 0$, and $\dot{\theta} = \omega$, it follows from Eq. (1) that the radius R is the solution of

$$R\omega^2 - \mu/R^2 - (\tfrac{3}{2})\mu J_2(R_e^2/R^4) = 0 \quad (2)$$

where ω is the Earth spin rate.

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The principal effect of the oblateness term J_2 is to increase the primary field stationary radius to the value R defined by Eq. (2). In order to simplify the analysis the following approximation is made

$$\mu/r^2 + \tfrac{3}{2}\mu J_2(R_e^2/r^4) = (\mu/r^2)[1 + \tfrac{3}{2}J_2R_e^2/(R+\delta r)^2] \cong (\mu/r^2)[1 + \tfrac{3}{2}J_2R_e^2/R^2] = \mu^*/r^2 \quad (3)$$

The binomial expansion shows that for $|\delta r| \leq 6$ km the error in the bracket caused by this approximation is less than that which corresponds to the discrepancy between the value of $J_2 = 1082.48 \times 10^{-6}$ (Ref. 1) and $J_2 = 1082.36 \times 10^{-6}$ (Ref. 2). Since the nominal orbit is circular and equatorial, the variational equation in the binormal direction is independent of the motion in the equatorial plane. This transversal motion, which is controllable by impulsive thrusting or compensated by biasing the initial orbit inclination with respect to the Earth equator, is not considered.

The geographic longitude of the satellite at time t is (see Fig. 1)

$$\lambda(t) = \lambda_0 + \Delta\lambda(t) \qquad \Delta\lambda(t) = \theta(t) - (t - t_0)\omega \quad (4)$$

where λ_0 is the station geographic longitude.

Longitude is defined as positive when east from Greenwich. Thus,

$$\lambda(t) - \lambda_2^{(2)} = \lambda_0 - \lambda_2^{(2)} + \Delta\lambda(t) \quad (5)$$

If

$$\zeta_0 = 90° + \lambda_0 - \lambda_2^{(2)} \quad (6)$$

then using Taylor's Theorem yields

$$\begin{aligned} \cos 2(\lambda - \lambda_2^{(2)}) &\cong -\cos 2\zeta_0 + 2\Delta\lambda(t)\sin 2\zeta_0 \\ \sin 2(\lambda - \lambda_2^{(2)}) &\cong -\sin 2\zeta_0 - 2\Delta\lambda(t)\cos 2\zeta_0 \end{aligned} \quad (7)$$

III. Closed Form Solution for the Motion under Perturbative and Low-Thrust Control Forces

The system of Eqs. (1) is now cast into an initial value problem relative to the time scale $\tau = N_1(t - t_1)$ (where N depends on the initial conditions at the initial time t_1 and is defined later);

$$\begin{aligned} dx_i/d\tau &= (1/N_1)g_i(x_j) + (\gamma/N_1)f_i(x_j,\tau) \qquad (i,j = 1,2,3,4) \\ x_i(t_1) &= a_i \end{aligned} \quad (8)$$

(where t_1 is the initial time) with nondimensional variables

$$\begin{aligned} x_1 &= r/R - 1,\ x_2 = \theta - \omega(t - t_0) = \Delta\lambda \\ x_3 &= \dot{r}/R\omega,\ x_4 = \dot{\theta}/\omega - 1 = \Delta\dot{\lambda}/\omega \end{aligned} \quad (9)$$

where $\gamma = A_0/R\omega^2$ is a nondimensional parameter and $A_0 = F_0/m$ is the reference thrust acceleration.

The functions g_i and f_i are defined as

$$\begin{aligned} g_1 &= \omega x_3,\ g_2 = \omega x_4 \\ g_3 &= \omega(x_1 + 1)(x_4 + 1)^2 - \omega/(x_1 + 1)^2 \\ g_4 &= -2\omega x_3(x_4 + 1)/(x_1 + 1) \\ f_1 &= 0,\ f_2 = 0 \\ f_3 &= B/(x_1 + 1)^4 + Cx_2/(x_1 + 1)^4 + uK_r(\tau)\omega \\ f_4 &= D/(x_1 + 1)^5 + Ex_2/(x_1 + 1)^5 + uK_\theta(\tau)\omega/(x_1 + 1) \end{aligned} \quad (10)$$

where ω, μ^*, R, B, C, D, E are constants and $u \gtrless 0$ is a control scalar, piecewise constant in time,

$$B = -9\mu J_2^{(2)}R_e^2 \cos 2\zeta_0\omega/R^4A_0 \quad (11)$$

$$C = 18\mu J_2^{(2)}R_e^2 \sin 2\zeta_0\omega/R^4A_0 \quad (12)$$

$$D = -6\mu J_2^{(2)}R_e^2 \sin 2\zeta_0\omega/R^4A_0 \quad (13)$$

$$E = -12\mu J_2^{(2)}R_e^2 \cos 2\zeta_0\omega/R^4A_0 \quad (14)$$

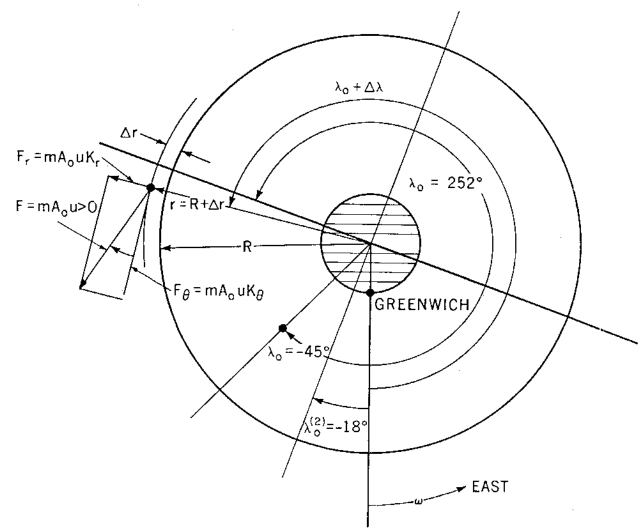

Fig. 1 Polar coordinates in the equatorial plane (North Polar axis out of paper).

When pitch motion is allowed K_r and K_θ are functions of time defined by

$$\begin{aligned} K_r(\tau) &= \sin[\alpha_0 + \beta_0 \sin(\Omega\tau/N_1 + \delta_0)] \\ K_\theta(\tau) &= \cos[\alpha_0 + \beta_0 \cos(\Omega\tau/N_1 + \delta_0)] \end{aligned} \quad (15)$$

where β_0, δ_0 are the amplitude and phase that characterize the linearized attitude pitch motion under gravity gradient stabilization. The pitch motion angular speed Ω is generally of the order of ω, $1.3\omega \leq \Omega \leq 1.5\omega$. The angle α_0 is the angle of the thrust force with the local horizontal in the zero pitch angle attitude.

Instead of treating the full Eqs. (8) the problem of solving

$$dx_i/d\tau = g_i(x_j)/N_1 \qquad (i,j = 1.4),\ x_i(t_1) = y_i \quad (16)$$

is considered first.

In order to initiate a variation of parameters solution[4] of Eqs. (8), the initial values of the x_i are denoted as y_i. Following the method of Lindstedt and Poincaré, which provides for a dependence of the period of the solution upon the initial values of the dependent variables, Fang and Brown[3] have obtained a first-order approximation to the solution of Eqs. (16) of the form

$$x(\tau) = \begin{pmatrix} 0 \\ \omega G_1\tau/N_1 \\ 0 \\ G_1/N_1 \end{pmatrix} + T(\tau)\begin{pmatrix} y_1 \\ y_2 \\ N_1y_3 \\ N_1y_4 \end{pmatrix} \quad (17)$$

where

$$\tau = (t - t_1)N_1 \qquad N_1 = 1 + G_1 \qquad G = -(6y_1 + 3y_4) \quad (18)$$

$$T(\tau) = \begin{pmatrix} 4 - 3\cos\omega\tau & 0 & \sin\omega\tau & 2(1 - \cos\omega\tau) \\ 6\sin\omega\tau & 1 & 2(\cos\omega\tau - 1) & 4\sin\omega\tau) \\ 3\sin\omega\tau & 0 & \cos\omega\tau & 2\sin\omega\tau) \\ 6\cos\omega\tau & 0 & -2\sin\omega\tau & 4\cos\omega\tau) \end{pmatrix} \quad (19)$$

Approximating Eq. (17) by neglecting squares and crossproducts of y_1 and y_4 yields

$$x(\tau) = \Phi(\tau)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \tag{20}$$

$$\Phi(\tau) = \begin{pmatrix} 4 - 3\cos\omega\tau & 0 & \sin\omega\tau & 2(1-\cos\omega\tau) \\ 6(\sin\omega\tau - \omega\tau) & 1 & 2(\cos\omega\tau - 1) & 4\sin\omega\tau - 3\omega\tau \\ 3\sin\omega\tau & 0 & \cos\omega\tau & 2\sin\omega\tau \\ 6(\cos\omega\tau - 1) & 0 & -2\sin\omega\tau & 4\cos\omega\tau - 3 \end{pmatrix}$$

Eqs. (20) defining the intermediary orbit, are now considered as equations for transforming Eqs. (8) from the original variables x_i to the arbitraries y_i.

Substituting Eqs. (20) in Eqs. (8) and using Eqs. (16) it follows that

$$\Phi(\tau)d\mathbf{y}(\tau)/d\tau + (d\Phi/d\tau)\mathbf{y} = (1/N_1)\mathbf{g} + (1/N_1)\gamma\mathbf{f} \tag{21}$$

Since $\Phi^{-1}(\tau) = \Phi(-\tau)$, as is easily verified, it follows from Eqs. (16) and (21) that

$$d\mathbf{y}/d\tau = (\gamma/N_1)\Phi(-\tau)\mathbf{f} \tag{22}$$

Hence, the variation of parameter equations are

$$dy_1/dt = \gamma\{-f_3\sin\omega\tau + f_4\,2(1-\cos\omega\tau)\} = \gamma F_1(y_j,t) \quad (j = 1,2,3,4)$$

$$dy_2/dt = \gamma\{f_3\,2(\cos\omega\tau - 1) - f_4(4\sin\omega\tau - 3\omega\tau)\} = \gamma F_2(y_j,t) \tag{23a}$$

$$dy_3/dt = \gamma\{f_3\cos\omega\tau - f_4\,2\sin\omega\tau\} = \gamma F_3(y_j,t) \tag{23b}$$

$$dy_4/dt = \gamma\{f_3\,2\sin\omega\tau + f_4(4\cos\omega\tau - 3)\} = \gamma F_4(y_j,t) \tag{23c}$$

$$y_1(t_1) = a_1 = r_1/R - 1,\ y_2(t_1) = a_2 = \Delta\lambda_1 \tag{23d}$$

$$y_3(t_1) = a_3 = \dot{r}_1/R\omega,\ y_4(t_1) = a_4 = \dot{\theta}_1/\omega - 1 = \Delta\dot{\lambda}_1/\omega \tag{23e}$$

where τ is defined by Eq. (18) and f_3, f_4 by Eqs. (10). The solution of Eqs. (23) is constructed in the form[4]

$$y_i(t) = a_i + y_i^{(1)}(t)\gamma + y_i^{(2)}\gamma^2 + + y_i^{(m)}\gamma^m \tag{24}$$

Hence, substituting Eq. (24) in Eqs. (23) and equating powers of γ and γ^2 results in

$$dy_i^{(1)}/dt = F_i(a_j,t) \tag{25}$$

$$\frac{dy_i^{(2)}}{dt} = \sum_{j=1}^{4}\frac{\partial F_i}{\partial y_j}y_j^{(1)} \tag{26}$$

The solution of Eq. (22) to order γ^2 is then obtained by the quadrature of Eqs. (25) and (26) which are closed form expressions given in the Appendix.†

The variation of parameters solution to Eq. (8) is therefore

$$\mathbf{x}(t_k) = \Phi(\tau_k)\begin{pmatrix} (r_j/R - 1) + \gamma y_1^{(1)}(\tau_k) + \gamma^2 y_1^{(2)}(\tau_k) \\ \Delta\lambda_j + \gamma y_2^{(1)}(\tau_k) + \gamma^2 y_2^{(2)}(\tau_k) \\ \dot{r}_j/R + \gamma y_3^{(1)}(\tau_k) + \gamma^2 y_3^{(2)}(\tau_k) \\ \Delta\dot{\lambda}_j/\omega + \gamma y_4^{(1)}(\tau_k) + \gamma^2 y_4^{(2)}(\tau_k) \end{pmatrix} =$$

$$\Phi(\tau_k)[\mathbf{x}_j + \gamma\mathbf{y}_j^{(1)}(\tau_k) + \gamma^2\mathbf{y}_j^{(2)}(\tau_k)],\ (k = j+1) \tag{27}$$

where the vectors $\mathbf{y}^{(1)}(\tau_k)$, $\mathbf{y}^{(2)}(\tau_k)$ are defined in the Appendix.

A rather complete list of solutions to this kind of low-thrust problem with $\mathbf{x}_o = 0$ is given in Ref. 5.

† It is assumed that K_r and K_θ are constants independent of time. When the pitch oscillation is larger than 5°, it may be necessary to introduce Eqs. (15) in Eqs. (10) and in the quadratures of Eqs. (25) and (26).

The solution given here is readily applicable to the station keeping guidance problem where generally $\mathbf{x}_o \neq 0$. Moreover, the introduction of the $\mathbf{x}_o$ dependent time scale τ (t) generates a better approximation for long term predictions of the state of motion.

In the case: $\gamma = 0.2743 \times 10^{-6}$, $|u| = 4$, $A_0 = F_0/m = 2.01077 \times 10^{-7}$ ft sec^{-2}, direct numerical comparisons have shown that Eq. (27) predicts the motion with sufficient accuracy for time intervals not exceeding 80 days.

In the particular case $\omega\tau = 2\pi$, $x_{10} = x_{20} = x_{20} = x_{40} = 0$, $u = 0$, Eq. (27) to order γ, reduces to Eqs. (32) and (31) of Frick and Garber.[6]

IV. Minimum Fuel Station Keeping Control Law

In attempting to formulate an optimum control problem for station keeping in terms of the nonlinear system defined by Eqs. (8) and (10), it is easily recognized that within the current status of optimal control theory[7] there is insight, but no completely sufficient constructive techniques to synthesize an optimal control law. For example, there are no known necessary or sufficient conditions to establish the complete controllability[8] of a given state $\mathbf{x}_0$ (i.e., the possibility of driving $\mathbf{x}_0$ to the origin in some finite time). Similarly, there are no available a priori conditions to investigate the sharper property of normality[9] for such nonlinear control problems.

As is known, normality implies the necessary structural characterization of the control sequence (bang-off-bang) for the minimum fuel regulator problem with constrained time of arrival to a desired target set.[7]

In principle, one could think of parameterizing the initial guess of the costate, and for each initial state numerically solve the state and costate differential equations in a search for the costate for which all the necessary conditions are fulfilled. Then, if the minimum fuel solution exists, and the fuel problem is of normal type, the switching times could be determined.[7] The solution to the costate equations adjoint to Eqs. (8) are then an almost periodic function of period close to 12 hr, with a switch (on) and (off) every half day. From system design considerations such frequent switching is not admissible. Moreover, it is known from simple calculations, that the sum total of the thrust (on) time intervals should be many days, in order to regulate the effect of initial condition errors and of the $J_2^{(2)}$ forces. Let us now form equations of smooth sampled motion, in which the daily periodic variations are ignored.

From Eq. (27) the instantaneous state is

$$\mathbf{x}(\tau) = \Phi(\tau)[\mathbf{x}_0 + \gamma\mathbf{y}^{(1)}(\tau) + \gamma^2\mathbf{y}\mathrm{f}^{2)}(\tau)] \tag{28}$$

The sampled state $\mathbf{x}_s(\tau)$ is now defined as that function of τ which results from setting $\omega\tau = 2\pi$ in the periodic and Poisson terms of Eqs. (28)

Hence

$$\mathbf{x}_s(\tau) = \Phi_s(\tau)[\mathbf{x}_0 + \gamma\mathbf{y}_s^{(1)}(\tau) + \gamma^2\mathbf{y}_s^{(2)}(\tau)] \tag{29}$$

is an expression that represents the perturbed state taken at intervals of 2π, where

$$\Phi_s(\tau) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -6\omega\tau & 1 & 0 & -3\omega\tau \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{y}_s^{(1)} = \begin{pmatrix} 2p_\theta\tau/N_0 \\ -2p_r\tau/N_0 + \frac{3}{2}p_\theta\omega\tau^2/N_0 \\ 0 \\ -3p_\theta\tau/N_0 \end{pmatrix} \tag{30}$$

By differentiation of Eq. (29) with respect to τ and by eliminating $\mathbf{x}_0$, the equation of motion for the sampled state is

$$\dot{\mathbf{x}}_s(\tau) = A_s\mathbf{x}_s + \Phi_s(\tau)[\gamma\dot{\mathbf{y}}_s^{(1)}(\tau) + \gamma^{2\cdot}\ ^{(2)}(\tau)]$$

$$A_s = \dot{\Phi}_s\Phi_s^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -6\omega & 0 & 0 & -3\omega \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (31)$$

and where the matrix A_s is independent of the perturbative and control forces.

Let the desired final sampled state be defined by $\mathbf{x}_{sd}$, and pose a minimum fuel regulator problem[7] with fixed response time τ_f and target state

$$\mathbf{x}_{sf} = \mathbf{x}_{sd} + \Delta\mathbf{x}_{pf}$$

for the system

$$\dot{\mathbf{x}}_s = A_s\mathbf{x}_s(\tau) + B\mathbf{u},\ \mathbf{x}_s(\tau_0) = \mathbf{x}_0,\ \mathbf{x}_s(\tau_f) = \mathbf{x}_f \quad (32)$$

where

$$B = \frac{\gamma\omega K_\theta}{N_0}\begin{pmatrix} 2 + C_1 & 0 & 0 & 0 \\ 0 & -2K_r/K_\theta + C_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & -3 + C_4 \end{pmatrix}$$

$$\mathbf{u} = u\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},\ \Delta\mathbf{x}_{pf} = -\frac{\gamma}{N_0}\int_0^{\tau_f}\begin{pmatrix} 2(D + x_{20}E) + d_1 \\ -2(B + x_{20}C) + d_2 \\ d_3 \\ -3(D + x_{20}E) + d_4 \end{pmatrix}d\tau \quad (33)$$

and where the $C_i(\tau)$ and $d_i(\tau)$ are introduced by terms in γ^2. To order γ, $C_i = d_i = 0$. Let

$$\mathbf{b} = \begin{pmatrix} 2 + C_1 \\ -2K_r/K_\theta + C_2 \\ C_3 \\ -3 + C_4 \end{pmatrix} \quad (34)$$

then Kalman's necessary and sufficient condition for complete controllability,[8,9]

$$\det(\mathbf{b}, A_s\mathbf{b}, A_s^2\mathbf{b}, A_s^3\mathbf{b}) \neq 0 \quad (35)$$

is not fulfilled by the sampled system with state Eq. (32). The sufficient conditions that characterize the optimal control sequence as bang-off-bang are then not applicable. Because of these difficulties in applying optimal control theory in a constructive form, the control problem is formulated in an ad hoc fashion which leads to an optimal switching sequence. In what follows control sequences are considered with two time intervals τ_1, τ_2 in which the control levels can assume any combination of the integers

$$\epsilon = -4, -3, -2, -1, 0, +1, +2, +3, +4 \quad (36)$$

with a discrete jump at the junction of τ_1 and τ_2. These thrust profiles include bang-bang, off-bang, and bang-off cases.

The general objective of the control sequence is to drive $\mathbf{x}_0$ to a desired final state $\mathbf{x}(\tau_2) = \mathbf{x}_f$. When Eq. (27) is applied to one of the sequences in Eq. (36) the final state becomes

$$\Phi(\tau_2)\{\Phi(\tau_1)[\mathbf{x}_0 + \mathbf{y}_0(\tau_1)] + \mathbf{y}_1(\tau_2)\} = \mathbf{x}_f \quad (37)$$

where the vectors

$$\mathbf{y}_j(\tau_k) = \gamma\mathbf{y}_j^{(1)}(\tau_k) + \gamma^2\mathbf{y}_j^{(2)}(\tau_k),\ k = j + 1,\ j = 0,1 \quad (38)$$

are defined by Eqs. (A1–A4) and Eqs. (A13–A16) of the Appendix.

For a given control sequence, the vectorial Eq. (37) implies four transcendental equations in two unknowns τ_1 and τ_2. If the periodic variations are ignored by formally setting $\omega\tau = 2\pi$, as before, then Eq. (37) becomes

$$\Phi_s(\tau_2)\{\Phi_s(\tau_1)[\mathbf{x}_0 + \mathbf{y}_{s0}(\tau_1)] + \mathbf{y}_{s1}(\tau_2)\} = \mathbf{x}_{sf} \quad (39)$$

a system of four algebraic equations in two unknowns. To order γ Eq. (39)‡ renders

$$x_{1f} = x_{10} + 2p_{\theta1}\gamma\tau_1 + 2p_{\theta2}\gamma\tau_2 \quad (40)$$

$$x_{2f} = x_{20} - (2p_{r1}\gamma + 6\omega x_{10} + 3\omega x_{40})\tau_1 - (2p_{r2}\gamma + 6\omega x_{10} + 3\omega x_{40})\tau_2 - 3\gamma\omega p_{\theta1}(\tau_1\tau_2 + \tau_1^2/2) - \tfrac{3}{2}\gamma\omega p_{\theta2}\tau_2^2 = x_{20}\epsilon + \varphi(\tau_1,\tau_2) \quad (41)$$

$$x_{3f} = x_{30} \quad (42)$$

$$x_{4f} = x_{40} - 3\gamma p_{\theta1}\tau_1 - 3\gamma p_{\theta2}\tau_2 = x_{40}\rho + \Psi(\tau_1,\tau_2) \quad (43)$$

where we have

$$x_{2f} = \epsilon x_{20} \quad (44)$$

$$x_{4f} = \rho x_{40} \quad (45)$$

while ϵ and ρ as chosen target state parameters, and where the divisors N_0, N_1 which are very close to unity have been neglected. It is evident that the component $x_3(\tau) \equiv x_{30} = \Delta r_0/R\omega$ is not affected by the controls, and that Eq. (43) and (40) are incompatible.

From Eqs. (40) and (43) it follows that in the order γ approximation

$$x_{1f} = x_{10} + \tfrac{2}{3}x_{40}(1 - \rho) = x_{10} + \tfrac{2}{3}(x_{40} - x_{4f})$$

Hence, the sampled state components $x_1 = \Delta r/R$ and $x_3 = \Delta\dot{r}/R\omega$ are not controllable, although both are affected by the controls in the γ^2 approximation.

The second and fourth of Eqs. (39) are compatible algebraic equations with a finite number of solutions. This is in agreement with the dynamical possibility that the initial state $\mathbf{x}_0$ may be driven to a final state $\mathbf{x}_f$ with a priori given components, x_{2f}, x_{4f} by control sequences of different duration $\tau_1 + \tau_2$.

The existence of such real positive solutions τ_1, τ_2 is tantamount to the state being partially controllable. Since it is possible to show that such solutions exist for x_0 in given neighborhoods and sufficiently strong control levels, the sampled state components x_2 and x_4 can then be regulated to zero. The iterative Newton-Raphson solution of the second and fourth of Eqs. (39) requires an accurate initial guess which can be determined on the basis of the order γ Eqs. (41) and (43). Although x_1 and x_3 cannot be regulated, they are affected by the controls and the final values of x_2 and x_4. At time $t_f = \tau_1/N_0 + \tau_2/N_1$ the osculating state $\mathbf{x}_f$ will characterize new values of osculating semimajor axis $R + ?a_f$ and eccentricity e_f. To assess the effect of the lack of controllability of the x_1, x_3 state components, a parameter

$$G_c = [(\delta a/R)^2 + W_e e^2] > 0 \quad (46)$$

is introduced and where $W_e > 0$, is an a priori adjustable weight factor. The parameter values G_{co} and G_{cf} are relative to the initial and final state, respectively. This parameter measures a weighted departure from the ideal circular sidereal orbit of radius R, and permits either the choice of control sequences that decrease it monotonically or the allowance of control sequences which do not increase it much.

A simple expression for G_c is given in the Appendix.

Since a minimum cost solution is desired an initial guess is determined for τ_1 and τ_2 by extremizing the cost of control

$$C = |u_1|\tau_1 + |u_2|\tau_2 = C(\tau_1,\tau_2) \quad (47)$$

with the constraints

$$x_{2f} - \epsilon x_{20} = \varphi(\tau_1,\tau_2) = 0 \quad (48)$$

$$x_{4f} - \rho x_{40} = \Psi(\tau_1,\tau_2) = 0 \quad (49)$$

‡ From now the sampled state is denoted without the subscript s.

which follow from Eqs. (41), (43–45). Let

$$F(\tau_1,\tau_2) = C(\tau_1,\tau_2) + \lambda\varphi(\tau_1,\tau_2) + \eta\Psi(\tau_1,\tau_2) \qquad (50)$$

where λ and η are Lagrange multipliers. The necessary conditions for a minimum

$$\partial F/\partial\tau_1 = 0 \qquad \partial F/\partial\tau_2 = 0 \qquad (51)$$

imply the linear nonhomogeneous system

$$\mathbf{b} + \lambda[\mathbf{c} + Q\tau] + \eta\mathbf{d} = 0, \mathbf{b} = \begin{pmatrix}|u_1|\\|u_2|\end{pmatrix}, \mathbf{d} = \begin{pmatrix}-3\gamma p_{\theta 1}\\-3\gamma p_{\theta 2}\end{pmatrix} \qquad (52)$$

where

$$Q = \begin{pmatrix}-3\gamma\omega p_{\theta 1} & -3\gamma\omega p_{\theta 1}\\-3\gamma\omega p_{\theta 1} & -3\gamma\omega p_{\theta 2}\end{pmatrix} \qquad (53)$$

The column vector $\mathbf{c}$ has components

$$c_i = -(2\gamma p_{ri} + 6\omega x_{10} + 3\omega x_{40}), (i = 1.2) \qquad (54)$$

The formal solution to Eq. (52) is a column vector of switching times

$$\tau = \begin{pmatrix}\tau_1\\\tau_2\end{pmatrix} = \frac{\mathbf{V}}{\lambda} + \mathbf{W} + \frac{\eta}{\lambda}\mathbf{z}, \mathbf{v} = -Q^{-1}\mathbf{b}$$

$$\mathbf{W} = -Q^{-1}\mathbf{c}, \mathbf{z} = -Q^{-1}\mathbf{d} \qquad (55)$$

Replacing Eq. (55) in Eq. (52) and by elinination of η one finds

$$\tau = \begin{pmatrix}\tau_1\\\tau_2\end{pmatrix} = \frac{1}{\lambda}(\mathbf{V} + \Omega_3\mathbf{z}) + \mathbf{W} + (\Omega_1 + \Omega_2)\mathbf{Z} \qquad (56)$$

$$\Omega_1 = x_{40}(1-\rho)/3\gamma(p_{\theta 1}z_1 + p_{\theta 2}z_2)$$
$$\Omega_2 = -(p_{\theta 1}W_1 + p_{\theta 2}W_2)/(p_{\theta 1}z_1 + p_{\theta 2}z_2) \qquad (57)$$
$$\Omega_3 = -(p_{\theta 1}V_1 + p_{\theta 2}V_2)/(p_{\theta 1}z_1 + p_{\theta 2}z_2)$$

Replacing Eq. (56) in Eq. (48) one obtains a quadratic

$$\Gamma_1\lambda^2 + \Gamma_2\lambda + \Gamma_3 = 0 \qquad (58)$$

where

$$\Gamma_1 = \Big\{x_{20}(1-\epsilon) - \sum_{j=1}^{2} W_j(2\gamma p_{rj} + 6\omega x_{10} + 3\omega x_{40}) - (3\gamma\omega p_{\theta 1})\left[W_2W_1 + \frac{W_1^2}{2}\right] - \frac{3}{2}\gamma\omega p_{\theta 2}W_2^2\Big\} \qquad (59)$$

$$\Gamma_2 = \Big\{-\sum_{j=1}^{2} V_j(2\gamma p_{rj} + 6\omega x_{10} + 3\omega x_{40}) - 3\gamma\omega p_{\theta 1}[V_1W_2 + V_2W_1 + V_1W_1] - 3\gamma\omega p_{\theta 2}V_2W_2\Big\} \qquad (60)$$

$$\Gamma_3 = \{-3\gamma\omega p_{\theta 1}[V_1V_2 + V_1^2/2] - \tfrac{3}{2}\gamma\omega p_{\theta 2}V_2^2\} \qquad (61)$$

With a given initial state $\mathbf{x}_0$, and given values of ϵ and ρ and for all sequences of controls, Eq. (58) is solved. Naturally, only those solutions $\tau_1^{(1)}$, $\tau_2^{(1)}$ given by Eq. (56) which are real positive are to be retained.

These initial guess solutions are improved into $\tau_1^{(2)}$, $\tau_2^{(2)}$ by application of the Newton-Raphson method to the second and fourth of Eqs. (39).

These $\tau_1^{(2)}$, $\tau_2^{(2)}$ solutions can be further improved by using the x_2 and x_4 equations of Eq. (37) which include periodic terms. In the last Newton-Raphson solution $\tau_1^{(3)}$, $\tau_2^{(3)}$, the time interval for the computation of partial derivatives is 36 times less than the interval of 6 hr used to obtain the $\tau_1^{(2)}$, $\tau^{(2)}$ solution.

By direct comparison we can then choose a control sequence on the one basis of its cost and its final effect on the orbit, which is measured by the value of the G_c parameter. In the computation of G_c as defined by Eq. (46) the final osculating state is used at time t_f, which is obtained by the recursive application of Eq. (27) at time intervals $\tau_1^{(3)}/20$ and $\tau_2^{(3)}/20$. This shorter arc recursive prediction increases the accuracy of the phase portraits $\Delta\dot{\lambda}$ vs $\Delta\lambda(t)$ and in the value of G_c. To map the sampled state vector $\mathbf{x}_s(t)$, Eq. (27) is used with $\omega\tau = 2\pi$ and the already described recursive fashion. This data is also used in isolating the optimum solution by verifying that the target state components x_{2f}, x_{4f} have been closely reached, and that adequate bounds for the state have not been violated from $t = t_0$ to $t = t_f$.

V. Numerical Results

The constants in the equation of motion are assumed to have the following values:

$$\mu = 3.986045981 \times 10^{14}m^3\,\text{sec}^{-2}$$
$$J_2 = 1082.48 \times 10^{-6}, J_2^{(2)} = -1.68 \times 10^{-6}$$
$$R = 42{,}164{,}851\text{m}, \lambda_2^{(2)} = -18°(\text{west})$$
$$\omega = 0.729211508\ 10^{-4}\ \text{sec}^{-1}$$
$$R_e = 6{,}378{,}166\text{ m}, F_0 = 5 \times 10^{-6}\text{ lb}, m = 24.866\text{ slugs}$$
$$A_0 = 2.01077 \times 10^{-7}ft\ \text{sec}^{-2}, \qquad \alpha_0 = 17°,$$
$$K_r = \sin\alpha_0, \qquad K_\theta = \cos\alpha_0$$

Regulation of Longitude and Longitude Rate Errors

Station longitude: $\lambda_0 = -45°$(west). Angle of thrust vector with respect to local horizontal: $\alpha_0 = 17°$, $\Delta r_0/R = x_{10} = 0$, $\Delta\lambda_0 = 3.6°$, $x_{20} = 6.2827 \times 10^{-2}$, $G_{co} = 0.0151 \times 10^{-6}$, $\Delta\dot{r}_0/R\omega = x_{30} = 0$, $\Delta\dot{\lambda}_0 = 0.0198°$/day, $x_{40} = 5.5 \times 10^{-5}$. Target substate; $\Delta\lambda_f = 0$, $\Delta\dot{\lambda}_f = 0$.

In this case there are two solutions.

1) Bang-bang solution $u_1 = +4$ (thrust directed towards east) $u_2 = -4$ (thrust directed towards west). The initial guess (order γ solution) is $\tau_1^{(1)} = 29$ days, 0 hr., 30 min, $\tau_2^{(1)} = 34$ days, 20 hr, 57 min. The Newton-Raphson solution to order γ^2 is $\tau_1^{(2)} = 13$ days, 18 hr, 6 min, $\tau_2^{(2)} = 15$ days, 21 hr, 39 min. The Newton-Raphson improvement of the last solution in the complete equations with the daily periodic terms, gives $t_1^{(3)} = 13$ days, 8 hr., 26 min., $t_2^{(3)} = 17$ days, 20 hr., 12 min. Final sampled state; $x_{1f} = 0.37 \times 10^{-4}$, $x_{2f} = -0.22 \times 10^{-8}$, $x_{3f} = 0.2 \times 10^{-6}$, $x_{4f} = -0.46 \times 10^{-10}$. Final osculating state; $x_{1f} = 0.24 \times 10^{-3}$, $x_{2f} = 0.2 \times 10^{-4}$, $G_{cf} = 0.4635 \times 10^{-6}$, $x_{3f} = 0.34 \times 10^{-4}$, $x_{4f} = -0.39 \times 10^{-3}$. The cost of control is $C_a \cong 128$ and $G_{cf}/G_{co} = 30.7$.

2) Off-bang solution $u_1 = 0$, $u_2 = -4$. The initial guess is $\tau_1^{(1)} \cong 138$ days, $\tau_2^{(1)} \cong 23$ days. The Newton-Raphson process required 4 iterations to converge to $\tau_1^{(2)} \cong 40$ days, $\tau_2^{(2)} \cong 4$ days and the cost of control is $C \simeq 16$. Final sampled state; $x_{1f} = 0.36 \times 10^{-4}$, $x_{2f} = 0.54 \times 10^{-8}$, $x_{3f} = 0.2 \times 10^{-6}$, $x_{4f} = 0.37 \times 10^{-9}$. Final osculating state; $x_{1f} = 0.25 \times 10^{-3}$, $x_{2f} - 0.19 \times 10^{-3}$, $x_{3f} = -0.46 \times 10^{-4}$, $x_{4f} = -0.44 \times 10^{-3}$, $G_{cf} = 0.11571 \times 10^{-6}$, $G_{cf}/G_{co} = 7.66$.

We observe that in case 1, $\tau_1 + \tau_2 \simeq 30$ days, $C_a = 128$; and in case 2,

$$\tau_1 + \tau_2 \simeq 44\text{ days}, C_b = 16 \cong 13\%\text{ of }C_a$$

The off-bang mode is definitely advantageous for these initial and final conditions;

$$C_b = 16 < C_a = 128, (G_{cf}/G_{co})_2 = 7.66 < (G_{cf}/G_{co})_1 = 30.7$$

Cyclic Station Keeping around the Origin of the $\Delta\lambda$, $\Delta\dot{\lambda}$ Phase Plane

The station is near the Pacific equilibrium point; $\lambda_0 = -108°$ (west). Each cyclic path around the origin was constructed as follows.

Table 1 Bang-off-bang cyclic control, $\lambda_0 = -108°$ (west)

$\Delta\lambda_0$	$\lvert\Delta\lambda'\rvert_{max}$	Cycle duration, days	Life time increase over bang-bang lifetime, %
1.5	1.9	105.33	57
1.00	1.4	86.64	43
0.75	1.2	76.60	34
0.5	0.80	66.05	25
0.25	0.65	54.86	13.6

Starting with common initial conditions; $x_{10} = 0$, $\Delta\lambda_0 = 3.6°$, $x_{30} = 0$, $\Delta\dot{\lambda}_0 = 0.02°$/day. Bang-bang and off-bang control sequences were generated for the final conditions $\Delta\lambda_f = 1.5°, 1°, 0.75°, 0.5°, 0.25°$, $\Delta\dot{\lambda}_f \equiv 0$. All the final values of Δr_f and $\Delta\dot{r}_f$ were very close to 1.6 km and 0.01 m/sec, respectively. With these states taken as initial states, control sequences were constructed which drove each $\Delta\lambda$ to its opposite value with $\Delta\dot{\lambda} \equiv 0$. The cycles were closed in a similar fashion, as shown in Fig. 2.

The maximum satellite lifetime decreases with the tightness of control about the station (see Table 1). If the initial error in altitude is assumed to be identically zero instead of about 1.6 km, the life times for bang-off-bang cyclic control increase about four times over the bang-bang lifetime. These results indicate the strong perturbative effect of initial orbital injection errors in altitude, altitude rate, and longitude rate.

The osculating state component $\Delta\dot{\lambda}$, is represented as a function of $\Delta\lambda$ for a time interval of two days in Fig. 3.

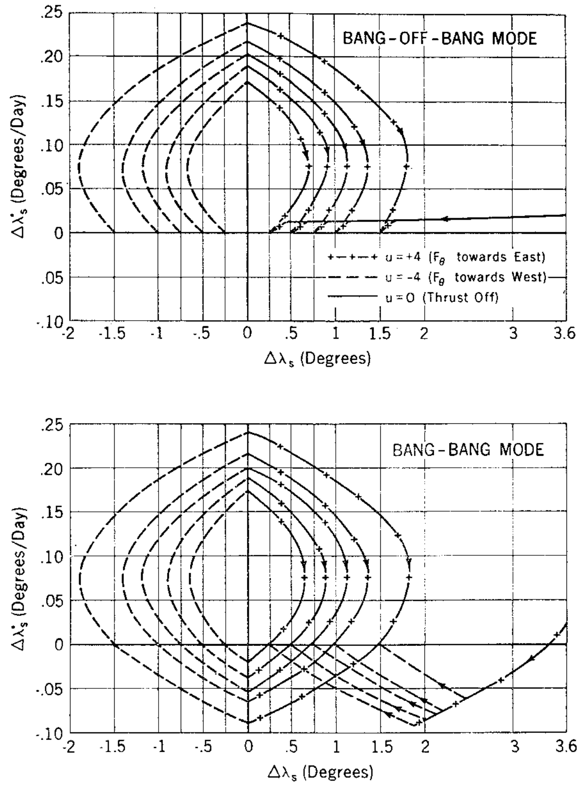

Fig. 2 Sampled state phase portraits for cyclic station keeping about $\lambda_o = -180°$ (west).

$$y_2^{(1)}(t) = (2p_r/\omega N)\sin\omega\tau + (4p_\theta/\omega N)\cos\omega\tau - (4p_\theta/\omega N) - 2p_r\tau/N + \tfrac{3}{2}p_\theta\omega\tau^2/N \quad \text{(A2)}$$

$$y_3^{(1)}(t) = (p_r/\omega N)\sin\omega\tau + (2p_\theta/\omega N)\cos\omega\tau - 2p_\theta/\omega N \quad \text{(A3)}$$

$$y_4^{(1)}(t) = (2p_r/\omega N)\cos\omega\tau + (4p_\theta/\omega N)\sin\omega\tau + 2p_r/\omega N - 3p_\theta\tau/N \quad \text{(A4)}$$

where the subscript 1 has been omitted in

$$N = 1 - 6\Delta r_1/R - 3\Delta\dot{\lambda}_1/\omega \quad \text{(A5)}$$

$$p_r = f_3(t_1) = BR^4/r_1^4 + C(R^4/r_1^4)\Delta\lambda_1 + u_1K_r\omega = p_{r1} \quad \text{(A6)}$$

$$p_\theta = f_4(t_1) = DR^5/r_1^5 + (ER^5/r_1^5)\Delta\lambda_1 + (R/r_1)u_1K_\theta\omega = p_{\theta 1} \quad \text{(A7)}$$

In order to construct the second-order approximation, the partial derivatives $(\partial F_i/\partial y_j)$ in Eqs. (36) are evaluated. From Eqs. (23) we are required to calculate $(\partial f_3/\partial y_j)_1$ and $(\partial f_4/\partial y_j)$. Hence, one finds

$$(\partial f_3/\partial y_1)_1 = p_{31} = -4BR^5/r_1^5 - 4CR^5\Delta\lambda_1/r_1^5 \quad \text{(A8)}$$

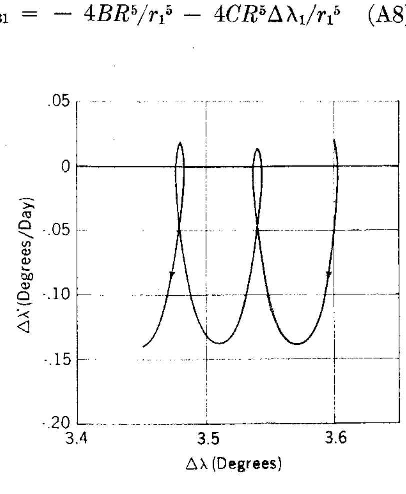

Fig. 3 Osculating state component $\Delta\dot{\lambda}$ as a function of $\Delta\lambda$, $\lambda_o = -108°$, $u = 0$.

Conclusion

A station keeping guidance scheme for a low-thrust gravity gradient stabilized 24-hr satellite that maximizes the lifetime has been presented. The motion consists of short period diurnal oscillations about a smooth motion which connects the states sampled in each revolution. The solution consists of the constant thrust levels and their direction, and the switching times for a given station and longitude excursion tolerance.

Numerical results show that relatively small initial errors in position and velocity, with respect to an idealized synchronous circular orbit, will cause long term longitude errors that require compensation by additional low thrusting. This demonstrates that a control law that ignores the important effect of initial errors in position and velocity will overestimate the lifetime of a low-thrust synchronous satellite.

The maximum lifetime decreases with the tightness of control about the station. Since the computations can be rapidly repeated on the basis of orbit determination data, this station keeping control scheme can be implemented in a discrete closed-loop fashion.

Luni-solar and higher harmonics gravitational perturbations and radiation pressure have been neglected. Because of their small order of magnitude, one conjectures that their effect on the motion can be compensated for by the closed-loop operation of the controls.

The lifetimes are longer for the near equilibrium longitude $\lambda_0 = -108°$(west) than for $\lambda_0 = -45°$(west). However, the lifetimes increase to a much greater extent with the closeness to synchronous circular orbit conditions at low-thrust initiation.

Appendix

Assuming K_r, K_θ piecewise constants in time and performing closed form quadratures of Eq. (25), we have, in terms of $\tau = N(t - t_1)$ where t_1 denotes any initial time;

$$y_1^{(1)}(t) = (p_r/\omega N)\cos\omega\tau - (2p_\theta/\omega N)\sin\omega\tau - (p_r/\omega N) + 2p_\theta\tau/N \quad \text{(A1)}$$

$$(\partial f_4/\partial y_1)_1 = p_{41} = -5DR^6/r_1^6 - 5ER^6/r_1^6 - \omega u_1 K_\theta R^2/r_1^6 \quad \text{(A9)}$$

$$(\partial f_3/\partial y_2)_1 = p_{32} = CR^4/r_1^4 \quad \text{(A10)}$$

$$(\partial f_4/\partial y_2)_1 = p_{42} = ER^5/r_1^5 \quad \text{(A11)}$$

$$(\partial f_3/\partial y_3) = (\partial f_4/\partial y_3) = (\partial f_3/\partial y_4) = (\partial f_4/\partial y_4) = 0 \quad \text{(A12)}$$

The quadrature of Eqs. (26) renders the γ^2 variations, with secular components;

$$y_1^{(2)}(\tau) = (p_r/\omega N)[-p_{41}\tfrac{3}{2}(\tau/N) - p_{32}{}^{3\tau} - 4p_{42}(\omega\tau^2/2 - \tau/N)] + p_\theta/\omega N[3p_{31}\tau/N + 4p_{41}(\tau/N - \omega\tau^2/2N) - 12p_{42}\tau/N + 3p_{32}(\omega\tau^2/2N - \tau/N) + 3p_{42}/N\ (\omega^2\tau^3/3 - \omega\tau^2 - 2\tau)] \quad \text{(A13)}$$

$$y_2^{(2)}(\tau) = (p_r/\omega N)[-p_{31}3\tau/N + 4p_{41}(\tau/N - \omega\tau^2/2N) - 12p_{42}\tau/N + 3p_{41}\tau/N - \tfrac{3}{2}p_{41}\omega\tau^2/N - 6p_{42}\tau/N - 2p_{42}\omega^2\tau^3/N] + (p_\theta/\omega N)[4p_{31}(\omega\tau^2/2N - \tau/N) + 12p_{41}\tau/N + 12p_{32}\tau/N + 6p_{41}\tau/N + 2p_{41}\omega^2\tau^3/N + 12p_{42}\tau/N - 6p_{42}\omega\tau^2/N - 3p_{32}\omega^2/N\ (\tau^3/3 - \tau^2/\omega - 2\tau/\omega^2) - 6p_{42}(\omega/N)(-\tau^2 + 2\tau/\omega) + \tfrac{9}{2}p_{42}\omega^3\tau^4/4N] \quad \text{(A14)}$$

$$y_3^{(2)}(\tau) = (p_r/\omega N)[p_{31}\tau/2N - 6p_{42}\tau/N] + (p_\theta/\omega N)[6p_{41}\tau/N + 5p_{32}\tau/N + 3p_{43}\omega\tau^2/N] \quad \text{(A15)}$$

$$y_4^{(2)}(\tau) = (p_r/\omega N)[5p_{41}\tau/N + 6p_{32}\tau/N + 3p_{42}\omega\tau^2/N] + (p_\theta/\omega N)[-6p_{31}\tau/N + 20p_{42}\tau/N - 3p_{32}\omega\tau^2/N + 12p_{42}\tau/N - \tfrac{3}{2}p_{42}\omega^2\tau^3/N] \quad \text{(A16)}$$

where the subscript 1 has been omitted.

Given the osculating state $\mathbf{x}(t)$, the parameter G_c in Eq. (46), is computed in terms of the corresponding eccentricity e and semimajor axis a by means of[10];

$$e^2 = \{[(v_\theta/v_c)^2 - 1]^2 + (v_\theta/v_c)^2(\dot{r}/v_c)^2\} \quad \text{(A17)}$$

$$v_\theta = R\omega(1 + x_1)(1 + x_4),\ v_c = [\mu^*/R(1 + x_1)]^{1/2} \quad \text{(A18)}$$

$$\dot{r} = R\omega x_3 \quad \text{(A19)}$$

$$a = \mu^*/\{2\mu^*/\mathrm{R}(1 + x_1) - R^2\omega^2[(1 + x_1)^2(1 + x_4)^2 + x_3^2]\} - R \quad \text{(A20)}$$

In Eq. (46) $W_e = 0.25$.

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